Chapter 3

Mathematical Induction

I have hardly ever known a mathematician who was capable of reasoning.

Plato (427 BC - 347 BC), The Republic

The integers , $1, 2, 3, 4, \ldots$ are also known as the natural numbers and *Mathematical induction* is a technique for proving a theorem, or a formula, that is asserted about every natural number. Suppose for example we believe

$$1 + 2 + 3 + ... + n = n(n + 1)/2$$

that is the sum of consecutive numbers from 1 to n is given by the formula on the right. We want to prove that this will be true for *all* n. As a start we can test the formula for any given number, say n = 3:

$$1 + 2 + 3 = 3 \times 4/2 = 6$$

It is also true for $n = 4$

$$1 + 2 + 3 + 4 = 4 \times 5/2 = 10$$

But how are we to prove this rule for every value of n? The method of proof we now describe is called the principle of mathematical induction. The idea is simple. Suppose we have some statement that is true for a particular natural number n and we want to prove that it is true for every value of n from $1, 2, 3, \ldots$ If all the following are true

1. When a statement is true for some natural number n, say k.
2. When it is also true for its successor, $k + 1$.
3. The statement is true for some value n, usually $n = 1$.

then the statement is true for every natural number n.

This is because, when the statement is true for $n = 1$, then according to 2, it will also be true for 2. But that implies it will be true for 3; which implies it will be true for 4. And so on. Hence it will be true for every natural number and thus is true for all n.

To prove a result by induction, then, we must prove parts 1, 2 and 3 above. The hypothesis of step 1

"The statement is true for $n = k$"

is called the induction assumption, or the induction hypothesis. It is what we assume when we prove a theorem by induction.

Example

Prove that the sum of the first n natural numbers is given by this formula:

$$S_n = 1 + 2 + 3 + ... + n = n(n + 1)/2$$

We will call this statement S_n, because it depends on n. Now we do steps 1 and 2 above.

1. First, we will assume that the statement is true for $n = k$ that is, we will assume that S_k is true so

   $$S_k = 1 + 2 + 3 + ... + k = k(k + 1)/2$$

   Note this is the induction assumption.

2. Assuming this, we must prove that $S_{(k+1)}$ is also true. That is, we need to show:

   $$S_{(k+1)} = 1 + 2 + 3 + ... + (k + 1) = (k + 1)(k + 2)/2$$

   To do that, we will simply add the next term $(k + 1)$ to both sides of the induction assumption,

   $$S_{(k+1)} = S_{(k+1)} + (k + 1) =$$

   $$1 + 2 + 3 + \ldots + (k + 1) = k(k + 1)/2 + (k + 1) = (k + 1)(k + 2)/2$$

   This is line 2, which is we wanted to show.

3. Next, we must show that the statement is true for $n = 1$. We have $S(1) = 1 = 1 \times 2/2$. The formula therefore is true for $n = 1$.

We have now fulfilled both conditions of the principle of mathematical induction. S_n is therefore true for every natural number.

Example

We prove that $8^n - 3^n$ is divisible by 5 for all $n \in \mathbb{N}$. The proof is by mathematical induction.

1. Assume the result holds for $n = k$, that is $8^k - 3^k \mod 5 = 0$. Then $8^{k+1} - 3^{k+1} = 8 \times 8^k - 3 \times 3^k$.

2. Now the clever step

   $$8^{k+1} - 3^{k+1} = 8 \times 8^k - 3 \times 3^k = 3 \times 8^k - 3 \times 3^k + 5 \times 8^k = 3 \times (8^k - 3^k) + 5 \times 8^k$$

   But $8^k - 3^k$ is divisible by 5 (by the induction hypothesis) and $5 \times 8k$ is obviously a multiple of 5. Therefore it follows that $(8^{k+1} - 3^{k+1})$ is divisible by 5. Hence, the result holds for $n = k + 1$.

3. The result holds for $n = 1$ because $8 - 3 = 5$ and so is divisible by 5.

So we have shown that the result holds for all n - by induction.

Another Example

We prove this rule of exponents:

$$(ab)^n = a^n b^n, \text{ for every natural number } n.$$

Call this statement $S(n)$ and assume that it is true when $n = k$; that is, we assume $S(k) = (ab)^k = a^k b^k$ is true.

We must now prove that $S(k+1)$ is true, that is

$$S(k+1) = (ab)^{k+1} = a^{k+1} b^{k+1}$$

Simply by multiplying both sides of line (3) by ab gives :

$$(ab)^k ab = a^k b^k ab = a^k ab^k b$$

since the order of factors does not matter,

$$(ab)^k ab = a^{k+1} b^{k+1}.$$

Which is what we wanted to show. So, we have shown that if the theorem is true for any specific natural number k, then it is also true for its successor, $k+1$. Next, we must show that the theorem is true for $n = 1$ which is trivial since $(ab)^1 = ab = a^1 b^1$.

This theorem is therefore true for every natural number n.

Exercises

In each of the following $0 \leq n$ is an integer

1. Prove that $n^2 + n$ is even.
2. Prove that $\sum_{i=1}^{n} n^2 = n(n+1)(2n+1)/6$.
3. Prove that $1 + 4 + 7 + \ldots + (3n-2) = n(3n-1)/2$.
4. Prove that $n! \geq 2^n$ when $n > 1$